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SOME PROPERTIES OF MITTAG-LEFFLER FUNCTIONS AND MATRIX-VARIATE ANALOGUES: A STATISTICAL PERSPECTIVE

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Abstract

Mittag-Leffler functions and their generalizations appear in a large variety of problems in different areas. When we move from total differential equations to fractional equations Mittag-Leffler functions come in naturally. Fractional reaction-diffusion problems in physical sciences and general input-output models in other disciplines are some of the examples in this direction. Some basic properties of Mittag-Leffler functions are examined first. Then representations in terms of Mellin-Barnes integrals are given, which are shown to yield many known and new results directly and easily. The results are presented in terms of statistical densities so that they are directly applicable to statistical distribution theory and stochastic processes. Several pathways are examined of exponential and gamma densities going to Mittag-Leffler densities and then Mittag-Leffler densities going to Lévy and Linnik densities. Then multivariable and matrix variable extensions of several results are given. Various results and representations given in this paper are directly applicable in many practical situations and are very suitable for further development of the theory. The material is presented in easily understandable formats, even for a beginner.

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1. Introductory Results on Mittag-Leffler Functions

We start with some introductory explorations of Mittag-Leffler distribution and its generalizations. Then we examine the Mellin-Barnes representations which give rise to some known and new results directly. Then we will extend to multivariate and then to matrix-variate cases. As it is well known that in physical systems when we move from the total differential or integral equations to the corresponding fractional versions then the solutions are usually available in terms of Mittag-Leffler functions and their generalizations. Thus a detailed study of Mittag-Leffler functions is very relevant.

This paper is organized as follows: Section 1 examines some basic results on Mittag-Leffler distributions. Sections 2 and 3 deal with generalized Mittag-Leffler functions and H-function representations. Section 4 gives structural representations. Section 5 examines various pathways. Section 6 deals with α-Laplace or Linnik distributions. Section 7 starts with multivariate extensions and then goes into matrix-variate extensions in the real case.

1.1. Mittag-Leffler statistical distribution and its properties

A statistical distribution in terms of the Mittag-Leffler function $E_\alpha(y)$ was defined by Pillai (1990) in terms of the distribution function or cumulative density function as follows:

$$G_y(y) = 1 - E_\alpha(-y^\alpha) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} y^{\alpha k}}{\Gamma(1+\alpha k)}, \quad 0 < \alpha \leq 1, y > 0 \tag{1.1}$$

and $G_y(y) = 0$ for $y \leq 0$. Differentiating on both sides with respect to y we obtain the density function $f(y)$ as follows:

$$\begin{aligned} f(y) &= \frac{\mathrm{d}}{\mathrm{d}y} G_y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left[\sum_{k=1}^{\infty} \frac{(-1)^{k+1} y^{\alpha k}}{\Gamma(1+\alpha k)}\right] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, \alpha k\ y^{\alpha k-1}}{\Gamma(1+\alpha k)} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} y^{\alpha k}}{\Gamma(\alpha k)} = \sum_{k=0}^{\infty} \frac{(-1)^{k} y^{\alpha+\alpha k-1}}{\Gamma(\alpha+\alpha k)}, \end{aligned}$$

by replacing k by $k+1$,

$$= y^{\alpha-1} E_{\alpha,\alpha}(-y^\alpha), \quad 0 < \alpha \leq 1,\ y > 0, \tag{1.2}$$

where $E_{\alpha,\beta}(x)$ is the generalized Mittag-Leffler function. Denoting $(\gamma)_k := \gamma(\gamma+1)...(\gamma+k-1),\ (\gamma)_0 = 1$, a more general Mittag-Leffler function is defined as follows:

$$E^{\gamma}_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k x^k}{k!\Gamma(\beta+\alpha k)}, \quad \gamma \neq 0,\ \alpha > 0,\ \beta > 0 \tag{1.3}$$

$$E^{1}_{\alpha,\beta}(x) = E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta+\alpha k)}, \quad \alpha > 0,\ \beta > 0 \tag{1.4}$$

$$E_{\alpha,1}(x) = E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\alpha k)}, \quad \alpha > 0; \quad E_1(x) = \mathrm{e}^x. \tag{1.5}$$

It is straightforward to observe that the Laplace transform of the density in (1.2) has the following form:

$$L_{E_{\alpha,\alpha}}(t) = \int_0^{\infty} \mathrm{e}^{-tx} x^{\alpha-1} E_{\alpha,\alpha}(-x^{\alpha})\mathrm{d}x = (1+t^{\alpha})^{-1}, \quad |t^{\alpha}| < 1. \tag{1.6}$$

Note that (1.6) is a special case of the general class of Laplace transforms discussed in Section 2.3.7 [Mathai and Haubold (2008)]. From (1.6) one can also note that there is a structural representation in terms of positive Lévy distribution. A positive Lévy random variable $u > 0$, with parameter α is such that the Laplace transform of the density of $u > 0$ is given by $\mathrm{e}^{-t^{\alpha}}$. That is,

$$E[\mathrm{e}^{-tu}] = \mathrm{e}^{-t^{\alpha}}, \tag{1.7}$$

where $E(\cdot)$ denotes the expected value of $(\cdot)$ or the statistical expectation of $(\cdot)$. When $\alpha = 1$ the random variable is degenerate with the density/ probability function

$$f_1(x) = \begin{cases} 1, \text{ for } x = 1 \\ 0, \text{ elsewhere.} \end{cases}$$

Consider a gamma random variable with the scale parameter δ and shape parameter β or with the density function

$$f_1(x) = \begin{cases} \dfrac{x^{\beta-1}\mathrm{e}^{-x/\delta}}{\delta^{\beta}\Gamma(\beta)}, \text{ for } 0 \leq x < \infty, \beta > 0, \delta > 0, \\ 0, \text{ elsewhere,} \end{cases} \qquad L_{f_1}(t) = (1+\delta t)^{-\beta}, \tag{1.8}$$

and with the Laplace transform $L_{f_1}(t)$. The following is a minor generalization of a known result. The proof is given by others by using different methods but we will obtain it by using conditional argument.

THEOREM 1.1. *Let $y > 0$ be a Lévy random variable with Laplace transform as in (1.7), $x \geq 0$ be a gamma random variable with the density as in (1.8) and let x and y be independently distributed. Then $u = yx^{\frac{1}{\alpha}}$ is distributed as a Mittag-Leffler random variable with Laplace transform*

$$L_u(t) = [1 + \delta t^{\alpha}]^{-\beta}. \tag{1.9}$$

P r o o f. For establishing this result, we will make use of a basic result on conditional expectations, which will be stated as a lemma.

LEMMA 1.1. *For two random variables x and y having a joint distribution,*

$$E(x) = E[E(x|y)], \tag{1.10}$$

whenever all the expected values exist, where the inside expectation is taken in the conditional space of x given y and the outside expectation is taken in the marginal space of y.

Now by applying (1.10) we have the following: Let the density of u be denoted by $g(u)$. Then the Laplace transform of g is given by

$$E[\mathrm{e}^{-(tx^{\frac{1}{\alpha}})y}|x] = \mathrm{e}^{-t^{\alpha}x}. \tag{1.11}$$

But the right side of (1.11) is in the form of a Laplace transform of the density of x with parameter t^{α}. Hence the expected value of the right side is

$$L_g(t) = (1 + \delta t^{\alpha})^{-\beta}, \quad \delta > 0, \beta > 0, \tag{1.12}$$

which establishes the result. From (1.11) one property is obvious. Suppose that we consider an arbitrary random variable y with the Laplace transform of the form

$$L_g(t) = \mathrm{e}^{-[\phi(t)]} \tag{1.13}$$

whenever the expected value exists, where $\phi(t)$ be such that

$$\phi(tx^{\frac{1}{\alpha}}) = x\phi(t), \lim_{t \to 0} \phi(t) = 0.$$

Then from (1.11) we have

$$E[\mathrm{e}^{-(tx^{\frac{1}{\alpha}})y}|x] = \mathrm{e}^{-x[\phi(t)]}. \tag{1.14}$$

Now, let x be an arbitrary positive random variable having Laplace transform, denoted by $L_x(t)$, where $L_x(t) = \psi(t)$. Then from (1.12) we have

$$L_g(t) = \psi[\phi(t)]. \tag{1.15}$$

For example, if y is a random variable whose density has the Laplace transform, denoted by $L_y(t) = \phi(t)$, with $\phi(tx^{\frac{1}{\alpha}}) = x\phi(t)$, and if x is a real random variable having the gamma density,

$$f_x(x) = \frac{x^{\beta-1}\mathrm{e}^{-\frac{x}{\delta}}}{\delta^{\beta}\Gamma(\beta)}, \quad x \geq 0, \beta > 0, \delta > 0, \tag{1.16}$$

and $f_x(x) = 0$ elsewhere, and if x and y are statistically independently distributed and if $u = yx^{\frac{1}{\alpha}}$, then the Laplace transform of the density of u, denoted by $L_u(t)$ is given by

$$L_u(t) = [1 + \delta\{\phi(t)\}]^{-\beta}. \tag{1.17}$$

NOTE 1.1. Since we did not put any restriction on the nature of the random variables, except that the expected values exist, the result in (1.15) holds whether the variables are continuous, discrete or mixed.

NOTE 1.2. Observe that for the result in (1.15) to hold we need only the conditional Laplace transform of y given x be of the form in (1.14) and the marginal Laplace transform of x be $\psi(t)$. Then the result in (1.15) will hold. Thus, statistical independence of x and y is not a basic requirement for the result in (1.15) to hold.

Thus, from (1.16) we may write a particular case, as

$$z = yx^{\frac{1}{\alpha}}, \tag{1.18}$$

where x is distributed as in (1.8) and y as in (1.7). Then z will be distributed as in (1.6) or (1.12) when x and y are assumed to be independently distributed.

NOTE 1.3. The representation of the Mittag-Leffler variable as well as the properties described on page 1432 of Jayakumar (2003) and on page 53 of the review paper of Jayakumar and Suresh (2003) are to be rewritten and corrected because the exponential variable and Lévy variable seem to be interchanged there. General properties of Mittag-Leffler functions, their generalizations, their applications to fractional calculus, reaction-diffusion type problems and so on, may be seen from many papers, for example, Gorenflo and Mainardi (1996), Gorenflo et al. (1998), Gorenflo et al.(2002), Kilbas (2005), and Kilbas and Saigo (1996).

By taking the natural logarithms on both sides of (1.18) we have

$$\frac{1}{\alpha}\ln x + \ln y = \ln z. \tag{1.19}$$

Then the first moment of $\ln z$ is available from (1.19) by computing $E[\ln x]$ and $E[\ln y]$. But $E[\ln x]$ is available from the following procedure: For positive real variables

$$E[\mathrm{e}^{-t\ln x}] = E[\mathrm{e}^{\ln x^{-t}}] = E[x^{-t}],$$

and in the exponential case:

$$= \int_0^\infty x^{-t}\mathrm{e}^{-x}\mathrm{d}x = \Gamma(1-t) \quad \text{for} \quad \Re(1-t) > 0, \tag{1.20}$$

which will be $\Gamma(\beta - t)/\Gamma(\beta)$ for the density in (1.16). Hence,

$$E[\ln x] = -\frac{\mathrm{d}}{\mathrm{d}t}E[\mathrm{e}^{-t\ln x}]|_{t=0} = -\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(1-t)|_{t=0}.$$

But

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(1-t) = -\Gamma(1-t)\frac{\mathrm{d}}{\mathrm{d}t}\ln\Gamma(1-t) = -\Gamma(1-t)\psi(1-t), \tag{1.21}$$

where $\psi(\cdot)$ is the psi function of $(\cdot)$, see Mathai (1993) for details. Hence by taking the limits $t \to 0$,

$$E[\ln x] = -\Gamma(1-t)\psi(1-t)|_{t=0} = -\psi(1) = \gamma, \tag{1.22}$$

where γ is Euler's constant, see Mathai (1993) for details.

2. Mellin-Barnes representation of the Mittag-Leffler density

Consider the density function in (1.2). After writing in series form and then looking at the corresponding Mellin-Barnes representation we have the following:

$$g(x) = x^{\alpha-1}E_{\alpha,\alpha}(-x^\alpha) = \sum_{k=0}^\infty \Gamma(1+k)\frac{(-1)^k}{k!}\frac{x^{\alpha-1+\alpha k}}{\Gamma(\alpha+\alpha k)} \tag{2.1}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\alpha}\frac{\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})\Gamma(1-\frac{1}{\alpha}+\frac{s}{\alpha})}{\Gamma(1-s)}x^{-s}\mathrm{d}s, \quad 1-\alpha < c < 1 \tag{2.2}$$

(by expanding as the sum of residues at the poles of $\Gamma(1-\frac{1}{\alpha}+\frac{s}{\alpha})$)

$$= \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\alpha s)}x^{\alpha s-1}\mathrm{d}s = g(x), \quad 0 < c_1 < 1, 0 < \alpha \le 1, \tag{2.3}$$

by putting $\frac{1}{\alpha} - \frac{s}{\alpha} = s_1$. By taking the Laplace transform of $g(x)$ from (2.1), we can verify that

$$L_g(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha + \alpha k)} \int_0^{\infty} x^{\alpha+\alpha k-1} \mathrm{e}^{-tx} \mathrm{d}x$$

$$= \sum_{k=0}^{\infty} (-1)^k t^{-\alpha-\alpha k} = t^{-\alpha}(1 + t^{-\alpha})^{-1} = (1 + t^{\alpha})^{-1}, \quad |t^{\alpha}| < 1. \tag{2.4}$$

2.1. Generalized Mittag-Leffler density

Consider the generalized Mittag-Leffler function

$$g_1(x) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\gamma + k)}{k! \Gamma(\alpha k + \alpha\gamma)} x^{\alpha\gamma - 1 + \alpha k}$$

$$= x^{\alpha\gamma-1} E_{\alpha\gamma,\alpha}^{\gamma}(-x^{\alpha}), \quad \alpha > 0, \gamma > 0. \tag{2.6}$$

The Laplace transform of $g_1(x)$ is as follows:

$$L_{g_1}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\gamma)_k}{\Gamma(\alpha\gamma + \alpha k)} \int_0^{\infty} x^{\alpha\gamma+\alpha k-1} \mathrm{e}^{-tx} \mathrm{d}x$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(\gamma)_k}{k!} t^{-\alpha\gamma-\alpha k} = (1 + t^{\alpha})^{-\gamma}, \quad |t^{\alpha}| < 1. \tag{2.7}$$

In fact, this is a special case of the general class of Laplace transforms connected with Mittag-Leffler function considered in Mathai et al. (2006).

3. Mittag-Leffler density as an H-function

The representations in terms of Mellin-Barnes integrals seem to yield very interesting results. Many known results can be derived directly and new results can be obtained through this procedure. $g_1(x)$ of (2.6) can be written as a Mellin-Barnes integral and then as an H-function:

$$g_1(x) = \frac{1}{\Gamma(\eta)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\eta - s)}{\Gamma(\alpha\eta - \alpha s)} x^{\alpha\eta-1} (x^{\alpha})^{-s} \mathrm{d}s, \ \Re(\eta) > 0, 0 < c < \Re(\eta),$$

$$= \frac{x^{\alpha\eta-1}}{\Gamma(\eta)} H_{1,2}^{1,1}\left[x^{\alpha} \Big|_{(0,1),(1-\alpha\eta,\alpha)}^{(1-\eta,1)} \right] \tag{3.1}$$

$$= \frac{1}{\alpha\Gamma(\eta)} \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma(\eta - \frac{1}{\alpha} + \frac{s}{\alpha})\Gamma(\frac{1}{\alpha} - \frac{s}{\alpha})}{\Gamma(1-s)} x^{-s} \mathrm{d}s \tag{3.2}$$

(by taking $\alpha\eta - 1 - \alpha s = -s_1$)

$$= \frac{1}{\alpha\Gamma(\eta)} H_{1,2}^{1,1}\left[x\Big|_{(\eta-\frac{1}{\alpha},\frac{1}{\alpha}),(0,1)}^{(1-\frac{1}{\alpha},\frac{1}{\alpha})}\right]. \tag{3.3}$$

Since g and g_1 are represented as inverse Mellin transforms, in the Mellin-Barnes representation, one can obtain the $(s-1)$-th moments of g and g_1 from (3.2). That is, with $M_{g_1}(s) = E(x^{s-1})$ in g_1,

$$= \frac{1}{\Gamma(\gamma)} \frac{\Gamma(\eta - \frac{1}{\alpha} + \frac{s}{\alpha})\Gamma(\frac{1}{\alpha} - \frac{s}{\alpha})}{\alpha\Gamma(1-s)}, \tag{3.4}$$

for $1 - \alpha < \Re(s) < 1, 0 < \alpha \le 1, \eta > 0$. If $M_g(t) = E(x^{s-1})$ in g, then

$$= \frac{\Gamma(1 - \frac{1}{\alpha} + \frac{s}{\alpha})\Gamma(\frac{1}{\alpha} - \frac{s}{\alpha})}{\alpha\Gamma(1-s)} \tag{3.5}$$

for $1 - \alpha < \Re(s) < 1, 0 < \alpha \le 1$, obtained by putting $\eta = 1$ also in (3.4). Since

$$\lim_{\alpha\to 1} \frac{\Gamma(\frac{1}{\alpha} - \frac{s}{\alpha})}{\Gamma(1-s)} = 1$$

for $\alpha \to 1$, (3.4) reduces to

$$M_{g_1}(t) = \frac{1}{\Gamma(\eta)}\Gamma(\eta - 1 + s) \quad \text{for } \alpha \to 1. \tag{3.6}$$

Its inverse Mellin transform is then

$$g_1 = \frac{1}{\Gamma(\eta)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\eta - 1 + s)x^{-s}\mathrm{d}s = \frac{1}{\Gamma(\eta)} x^{\eta-1}\mathrm{e}^{-x}, \quad x \ge 0, \eta > 0, \tag{3.7}$$

which is the one-parameter gamma density and for $\eta = 1$ it reduces to the exponential density. Hence the generalized Mittag-Leffler density g_1 can be taken as an extension of a gamma density such as the one in (3.7) and the Mittag-Leffler density g as an extension of the exponential density for $\eta = 1$. Is there a structural representation for the random variable giving rise to the Laplace transform in (2.4) corresponding to (1.13)? *The answer is in the affirmative* and it is illustrated in (1.17).

Note 3.1. Pillai (1990, Theorem 2.2), Lin (1998, Lemma 3) and others list the ρ-th moment of the Mittag-Leffler density g in (1.2) as follows, obtained through other procedures:

$$E(x^\rho) = \frac{\Gamma(1 - \frac{\rho}{\alpha})\Gamma(1 + \frac{\rho}{\alpha})}{\Gamma(1-\rho)}, \quad -\alpha < \Re(\rho) < \alpha < 1.$$

Therefore,

$$E(x^{s-1}) = \frac{\Gamma(1+\frac{1}{\alpha}-\frac{s}{\alpha})\Gamma(1-\frac{1}{\alpha}+\frac{s}{\alpha})}{\Gamma(2-s)}$$

$$= \frac{(\frac{1}{\alpha}-\frac{s}{\alpha})\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})\Gamma(1-\frac{1}{\alpha}+\frac{s}{\alpha})}{(1-s)\Gamma(1-s)} = \frac{1}{\alpha}\frac{\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})\Gamma(1-\frac{1}{\alpha}+\frac{s}{\alpha})}{\Gamma(1-s)},$$

which is the expression in (3.5). Hence the two expressions are one and the same.

Note 3.2. If $y = ax, a > 0$ and if x has a Mittag-Leffler distribution, then the density of y can also be represented as a Mittag-Leffler function with the Laplace transform

$$L_x(t) = (1+t^\alpha)^{-1} \Rightarrow L_y(t) = (1+(at)^\alpha)^{-1}, \quad a > 0, \, |(at)^\alpha| < 1.$$

Note 3.3. From the representation that

$$E(x^h) = \frac{\Gamma(1-\frac{h}{\alpha})\Gamma(1+\frac{h}{\alpha})}{\Gamma(1-h)}, \quad -\alpha < \Re(h) < \alpha < 1,$$

we have

$$E(x^0) = \frac{\Gamma(1)\Gamma(1)}{\Gamma(1)} = 1.$$

Further, $g(x)$ in (2.1) is a non-negative function for all x, with

$$E(x^h) = \int_0^\infty x^h g(x) \mathrm{d}x = 1 \quad \text{for} \quad h = 0.$$

Hence $g(x)$ is a density function for a positive random variable x. Note that from the series form for the Mittag-Leffler function it is not possible to show that $\int_0^\infty g(x)\mathrm{d}x = 1$ directly. There does not seem to be such a direct proof available in the literature. The usual procedure seems to be to show that (1.1) has all the properties of a statistical distribution function.

4. Structural representation of the generalized Mittag-Leffler variable

Let u be the random variable corresponding to the Laplace transform (2.7) with t^α replaced by δt^α and γ by η. Let u be a positive Lévy variable with the Laplace transform $\mathrm{e}^{-t^\alpha}, 0 < \alpha \leq 1$ and let v be a gamma random variable with parameters η and δ or with the Laplace transform

$(1+\delta t)^{-\eta}, \eta > 0, \delta > 0$. Let u and v be statistically independently distributed.

From the structural representation $w = uv^{\frac{1}{\alpha}}$, taking the Mellin transforms and writing as expected values, we have

$$E(w^{s-1}) = E(u^{s-1})E(v^{\frac{1}{\alpha}})^{s-1}, \tag{4.1}$$

due to statistical independence of u and v. The left side is available from (3.4) as

$$E(w^{s-1}) = \frac{\Gamma(\eta - \frac{1}{\alpha} + \frac{s}{\alpha})\Gamma(\frac{1}{\alpha} - \frac{s}{\alpha})\delta^{\frac{s-1}{\alpha}}}{\alpha\Gamma(\eta)\Gamma(1-s)}. \tag{4.2}$$

Let us compute $E[v^{\frac{1}{\alpha}}]^{s-1}$ from the gamma density. That is,

$$E[v^{\frac{1}{\alpha}}]^{s-1} = \frac{1}{\delta^{\eta}\Gamma(\eta)}\int_0^{\infty}(v^{\frac{1}{\alpha}})^{s-1}v^{\eta-1}\mathrm{e}^{-\frac{v}{\delta}}\mathrm{d}v = \frac{\Gamma(\eta - \frac{1}{\alpha} + \frac{s}{\alpha})\delta^{\frac{s-1}{\alpha}}}{\Gamma(\eta)} \tag{4.3}$$

for $\Re(s) > 1 - \alpha\eta,\ 0 < \alpha \leq 1, \eta > 0$. Comparing (4.2) and (4.3), we have the $(s-1)$-th moment of a Lévy variable

$$E[u^{s-1}] = \frac{\Gamma(\frac{1}{\alpha} - \frac{s}{\alpha})}{\alpha\Gamma(1-s)} = \frac{\Gamma(1 + \frac{1}{\alpha} - \frac{s}{\alpha})}{\Gamma(2-s)},\quad \Re(s) < 1, 0 < \alpha \leq 1. \tag{4.4}$$

NOTE 4.1. Lin (1998) gives the ρ-th moment of a Lévy variable with parameter α, through another procedure, as

$$E[u^{\rho}] = \frac{\Gamma(1 - \frac{\rho}{\alpha})}{\Gamma(1-\rho)}. \tag{4.5}$$

Hence for $\rho = s - 1$ we have

$$E[u^{s-1}] = \frac{\Gamma(1 + \frac{1}{\alpha} - \frac{s}{\alpha})}{\Gamma(2-s)} = \frac{(\frac{1}{\alpha} - \frac{s}{\alpha})\Gamma(\frac{1}{\alpha} - \frac{s}{\alpha})}{(1-s)\Gamma(1-s)} = \frac{\Gamma(\frac{1}{\alpha} - \frac{s}{\alpha})}{\alpha\Gamma(1-s)} \text{ for } \Re(s) < 1.$$

This is (4.4) and hence both the representations are one and the same. Observe that the representation in (4.4) also yields explicit form of the Lévy density as an inverse Mellin transform.

Hence the Lévy density, denoted by $g_2(u)$, can be written as

$$g_2(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\frac{1}{\alpha} - \frac{s}{\alpha})}{\alpha\Gamma(1-s)}u^{-s}\mathrm{d}s = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(1 + \frac{1}{\alpha} - \frac{s}{\alpha})}{\Gamma(2-s)}u^{-s}\mathrm{d}s. \tag{4.6}$$

We can easily verify its Laplace transform. Its Laplace transform is then:

$$L_{g_2}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})}{\alpha\Gamma(1-s)}[\int_0^\infty u^{1-s-1}\mathrm{e}^{-tu}\mathrm{d}u]\mathrm{d}s$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{\alpha}\Gamma(\frac{1}{\alpha}-\frac{s}{\alpha})t^{-1+s}\mathrm{d}s = \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} \frac{1}{\alpha}\Gamma(\frac{s}{\alpha})t^{-s}\mathrm{d}s, \qquad (4.7)$$

by making the substitution $-1+s=-s_1$. Then, evaluating as the sum of the residues at $\frac{s}{\alpha}=-\nu, \nu=0,1,2,...$ we have

$$L_{g_2}(t) = \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}t^{\alpha\nu} = \mathrm{e}^{-t^\alpha}. \qquad (4.8)$$

This verifies the result about the Laplace transform of the positive Lévy variable with parameter α. Note that when $\alpha=1$, (4.8) gives the Laplace transform of a degenerate random variable taking the value 1 with probability 1.

The Mellin convolution of certain Lévy variables can be seen to be again a Lévy variable.

LEMMA 4.1. *Let x_j be a positive Lévy variable with parameter $\alpha_j, 0<\alpha_j\le 1$ and let $x_1,...,x_p$ be statistically independently distributed. Then*

$$u = x_1 x_2^{\frac{1}{\alpha_1}}\dots x_p^{\frac{1}{\alpha_1\alpha_2\dots\alpha_{p-1}}}$$

is distributed as a Lévy variable with parameter $\alpha=\alpha_1\alpha_2\dots\alpha_p$.

P r o o f. From (4.4),

$$E[\mathrm{e}^{-tx_j}] = \mathrm{e}^{-t^{\alpha_j}}, \quad 0<\alpha_j\le 1,\ j=1,...,p$$

$$E[\mathrm{e}^{-tu}] = E[\mathrm{e}^{-tx_1x_2^{\frac{1}{\alpha_1}}\dots x_p^{\frac{1}{\alpha_1\dots\alpha_{p-1}}}}] = E[E[\mathrm{e}^{-tx_1\dots x_p^{\frac{1}{\alpha_1\dots\alpha_{p-1}}}}|_{x_2,\dots,x_p}]]$$

$$= E[\mathrm{e}^{-t^{\alpha_1}x_2x_3^{\frac{1}{\alpha_2}}\dots x_p^{\frac{1}{\alpha_2\dots\alpha_{p-1}}}}].$$

Repeated application of the conditional argument gives the final result as

$$E[\mathrm{e}^{-tu}] = \mathrm{e}^{-t^\alpha}, \quad \alpha=\alpha_1\alpha_2\dots\alpha_p, \quad 0<\alpha_1\dots\alpha_p\le 1,$$

which means that u is distributed as a Lévy with parameter $\alpha=\alpha_1\dots\alpha_p$.

From the representation $w=uv^{\frac{1}{\alpha}}$, we can compute the moments of the natural logarithms of Mittag-Leffler, Lévy and gamma variables,

$$w = uv^{\frac{1}{\alpha}} \Rightarrow \ln w = \ln u + \frac{1}{\alpha}\ln v. \qquad (4.9)$$

But from (4.2) and (4.5) we have the h-th moments of u and v given by

$$E[u^h] = \frac{\Gamma(1-\frac{h}{\alpha})}{\Gamma(1-h)}, \quad \Re(h) < \alpha \le 1, \tag{4.10}$$

and

$$E[v^{\frac{1}{\alpha}}]^h = \frac{\Gamma(\eta+\frac{h}{\alpha})\delta^{\frac{h}{\alpha}}}{\Gamma(\eta)}, \quad \Re(\eta+\frac{h}{\alpha}) > 0. \tag{4.11}$$

From (4.2),

$$E[w^h] = \frac{\Gamma(\eta+\frac{h}{\alpha})\Gamma(1-\frac{h}{\alpha})\delta^{\frac{h}{\alpha}}}{\Gamma(\eta)\Gamma(1-h)}. \tag{4.12}$$

But for a positive random variable z,

$$E[z^h] = E[\mathrm{e}^{\ln z^h}].$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}h}E[z^h]|_{h=0} = E[\ln z\mathrm{e}^{h\ln z}]_{h=0} = E[\ln z].$$

Therefore from (4.9) to (4.12), we have the following:

$$E[\ln w] = \frac{\mathrm{d}}{\mathrm{d}h}\left\{\frac{\delta^{\frac{h}{\alpha}}\Gamma(\eta+\frac{h}{\alpha})\Gamma(1-\frac{h}{\alpha})}{\Gamma(\eta)\Gamma(1-h)}\right\}|_{h=0} = \frac{1}{\alpha}\psi((\eta) - \frac{1}{\alpha}\psi(1) + \psi(1) + \frac{1}{\alpha}\ln\delta, \tag{4.13}$$

by taking the logarithmic derivative, where ψ is a psi function, see, for example Mathai (1993),

$$E[\ln v^{\frac{1}{\alpha}}] = \frac{\mathrm{d}}{\mathrm{d}h}\left\{\frac{\delta^{\frac{h}{\alpha}}\Gamma(\eta+\frac{h}{\alpha})}{\Gamma(\eta)}\right\}|_{h=0} = \frac{1}{\alpha}\psi(\eta) + \frac{1}{\alpha}\ln\delta, \tag{4.14}$$

or

$$E[\ln v] = \psi(\eta) + \ln\delta$$

and

$$E[\ln u] = \frac{\mathrm{d}}{\mathrm{d}h}\left\{\frac{\Gamma(1-\frac{h}{\alpha})}{\Gamma(1-h)}\right\}|_{h=0} = -\frac{1}{\alpha}\psi(1) + \psi(1), \tag{4.15}$$

where $\psi(1) = -\gamma$ and γ is the Euler's constant.

NOTE 4.2. The relations on the expected values of the logarithms of Mittag-Leffler variable, positive Lévy variable and exponential variable, given on page 1432 of Jayakumar (2003), where $\eta = 1$, are not correct.

5. A pathway from Mittag-Leffler variable to positive Lévy variable

Consider the function

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k(\eta)_k}{k!\Gamma(\alpha\eta+\alpha k)} \frac{x^{\alpha\eta-1+\alpha k}}{(a^{\frac{1}{\alpha}})^{\alpha\eta+\alpha k}}, \qquad \eta>0,\ a>0,\ 0<\alpha\le 1, \\ &= \frac{x^{\alpha\eta-1}}{a^\eta} E^{\eta}_{\alpha,\alpha\eta}(-\frac{x^\alpha}{a}). \end{aligned}$$

Thus $x = a^{\frac{1}{\alpha}} y$, where y is a generalized Mittag-Leffler variable. The Laplace transform of f is given by the following:

$$\begin{aligned} L_f(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k(\eta)_k}{k!a^{\eta+k}} \int_0^\infty \frac{x^{\alpha\eta+\alpha k-1}\mathrm{e}^{-tx}}{\Gamma(\alpha\eta+\alpha k)}\mathrm{d}x \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k(\eta)_k}{k!a^{\eta+k}} t^{-\alpha\eta-\alpha k} = [1+at^\alpha]^{-\eta}, \quad |at^\alpha|<1. \end{aligned} \tag{5.1}$$

If η is replaced by $\frac{\eta}{q-1}$ and a by $a(q-1)$ with $q>1$ then we have a Laplace transform

$$L_f(t) = [1+a(q-1)t^\alpha]^{-\frac{\eta}{q-1}}, \quad q>1. \tag{5.2}$$

If $q \to 1_+$, then

$$L_f(t) \to \mathrm{e}^{-a\eta t^\alpha} = L_{f_1}(t), \tag{5.3}$$

which is the Laplace transform of a constant multiple of a positive Lévy variable with parameter α. Thus q here creates a pathway of going from the general Mittag-Leffler density f to a positive Lévy density f_1 with parameter α, the multiplying constant being $(a\eta)^{\frac{1}{\alpha}}$. For a discussion of a general rectangular matrix-variate pathway model see Mathai (2005). The result in (5.3) can be put in a more general setting. Consider an arbitrary real random variable y with the Laplace transform, denoted by $L_y(t)$, and given by

$$L_y(t) = \mathrm{e}^{-\phi(t)}, \tag{5.4}$$

where $\phi(t)$ is a function such that $\phi(tx^\gamma) = x\phi(t), \phi(t) \ge 0, \lim_{t\to 0}\phi(t) = 0$ for some real positive γ. Let

$$u = yx^\gamma, \tag{5.5}$$

where x and y are independently distributed with y having the Laplace transform in (5.4) and x having a two-parameter gamma density with shape parameter β and scale parameter δ or with the Laplace transform

$$L_x(t) = (1+\delta t)^{-\beta}. \tag{5.6}$$

Now consider the Laplace transform of u in (5.5), denoted by $L_u(t)$. Then

$$L_u(t) = E[\mathrm{e}^{-tu}] = E[\mathrm{e}^{-tyx^\gamma}] = E[E[\mathrm{e}^{tyx^\gamma}|x]]$$

$$= E[\mathrm{e}^{-\phi(tx^\gamma)}] = E[\mathrm{e}^{-x\phi(t)}]$$

from the assumed property of $\phi(t)$,

$$= [1+\delta\phi(t)]^{-\beta}. \tag{5.7}$$

If δ is replaced by $\delta(q-1)$ and β by $\frac{\beta}{q-1}$ with $q>1$ then we get a path through q. That is, when $q \to 1_+$,

$$L_u(t) = [1+\delta(q-1)\phi(t)]^{-\frac{\beta}{q-1}} \to \mathrm{e}^{-\delta\beta\phi(t)} = \mathrm{e}^{-\phi(\delta^\gamma\beta^\gamma t)}. \tag{5.8}$$

If $\phi(t) = t^\alpha, 0 < \alpha \le 1$, then

$$L_u(t) = \mathrm{e}^{-(\delta\beta)^{\gamma\alpha}t^\alpha}$$

which means that u goes to a constant multiple of a positive Lévy variable with parameter α, the constant being $(\delta\beta)^\gamma$. The path from Mittag-Leffler to positive Lévy is described by the pathway parameter q here. The more general situation is coming from (5.7) to (5.8).

6. Linnik or α-Laplace distribution and multivariable analogues

A Linnik random variable is defined as that real scalar random variable whose characteristic function is given by

$$\phi(t) = \frac{1}{1+|t|^\alpha}, \quad 0 < \alpha \le 2, \ -\infty < t < \infty. \tag{6.1}$$

For $\alpha = 2$, (6.1) corresponds to the characteristic function of a Laplace random variable and hence Pillai and his coworkers name the distribution corresponding to (6.1) as the α-Laplace distribution. For positive variable, (6.1) reduces to the characteristic function of a Mittag-Leffler variable. Infinite divisibility, characterizations, other properties and related materials may be seen from the review paper Jayakumar and Suresh (2003) and the many references therein, Pakes (1998) and Mainardi and Pagnini (2008). Multivariate generalization of Mittag-Leffler and Linnik distributions may be seen from Lim and Teo (2009). Since the steps for deriving results on Linnik distribution are parallel to those of the Mittag-Leffler variable, further discussion of Linnik distribution is omitted. Linnik or α-Laplace distribution plays vital roles in non-Gaussian stochastic processes and time series.

A multivariate Linnik distribution can be defined in terms of a multivariate Lévy vector. Let $T' = (t_1, ..., t_p), X' = (x_1, ..., x_p)$, prime denoting the transpose. A vector variable having positive Lévy distribution is given by the characteristic function

$$E[\mathrm{e}^{iT'X}] = \mathrm{e}^{-(T'\Sigma T)^{\frac{\alpha}{2}}}, \quad 0 < \alpha \leq 2, \tag{6.2}$$

where $\Sigma = \Sigma' > 0$ is a real positive definite $p \times p$ matrix. Consider the representation

$$u = y^{\frac{1}{\alpha}} X, \tag{6.3}$$

where the $p \times 1$ vector X, having a multivariable Lévy distribution with parameter α, and y a real scalar gamma random variable with the parameters δ and β, are independently distributed. Then the characteristic function of the random vector variable u is given by the following:

$$E[\mathrm{e}^{iy^{\frac{1}{\alpha}} T'X}] = E[E[\mathrm{e}^{iy^{\frac{1}{\alpha}} T'X}]|_y]$$

$$= E[\mathrm{e}^{-y[|T'\Sigma T|]^{\frac{\alpha}{2}}}] = [1 + \delta |T'\Sigma T|^{\frac{\alpha}{2}}]^{-\beta}. \tag{6.4}$$

Then the distribution of u, with the characteristic function in (6.4) is called a vector-variable Linnik distribution. Some properties of this distribution are given in Lim and Teo (2009).

7. Matrix-variate analogues

Let us look into some limiting processes first. One can see that a central limit property can be established but it will be of the nature of generalized Mittag-Leffler variable going to a positive Lévy variable. Consider the generalized Mittag-Leffler density with the Laplace transform

$$L_x(t) = [1 + \delta t^{\alpha}]^{-\beta}, \quad \delta > 0, \beta > 0, 0 < \alpha \leq 1. \tag{7.1}$$

Let $x_j, j = 1, ..., n$ be independently and identically distributed as in (7.1) and let

$$y = x_1 + ... + x_n, \ u = \bar{x} = \frac{x_1 + ... + x_n}{n}, \ v = n^{\frac{\alpha - 1}{\alpha}} u = \frac{(x_1 + ... + x_n)}{n^{\frac{1}{\alpha}}}.$$

Then,

$$L_y(t) = [1 + \delta t^{\alpha}]^{-n\beta}, \tag{7.2}$$

or y is again a Mittag-Leffler variable with parameters δ and $n\beta$, and

$$L_v(t) = [1 + \frac{\delta t^\alpha}{n}]^{-n\beta}. \tag{7.3}$$

Hence when $n \to \infty$,

$$\lim_{n\to\infty} L_v(t) = \mathrm{e}^{-\delta\beta t^\alpha} \tag{7.4}$$

which is the Laplace transform of a constant multiple of a positive Lévy variable with parameter α, the constant being $(\delta\beta)^{\frac{1}{\alpha}}$. Thus the central limiting property is that a certain normalized sample mean from a Mittag-Leffler population goes to a Lévy variable.

If we replace δ by $\delta(q-1)$ and β by $\frac{\beta}{q-1}, q > 1$ then as $q \to 1$,

$$\lim_{q\to 1}[1 + \delta(q-1)t^\alpha]^{-\frac{\beta}{q-1}} = \mathrm{e}^{-\delta\beta t^\alpha}, \tag{7.5}$$

where q is a pathway parameter describing the path of going from a general Mittag-Leffler variable to a constant multiple of a Lévy variable. Now, we look into some matrix-variable analogues.

Let $X = (x_{rs})$ be $m \times n$, where all x_{rs}'s are distinct, X be of full rank, and having a joint density $f(X)$. Then the characteristic function of $f(X)$, denoted by $\phi_X(T)$, is given by

$$\phi_X(T) = E[\mathrm{e}^{i\mathrm{tr}(XT)}], \quad i = \sqrt{-1} \;\text{ and }\; T = (t_{rs}) \tag{7.6}$$

is an $n \times m$ matrix of distinct parameters t_{rs}'s and let T be of full rank.

Example 1. Consider the real matrix-variate Gaussian density

$$g(X) = \frac{|A|^{\frac{n}{2}}|B|^{\frac{m}{2}}}{\pi^{\frac{mn}{2}}}\mathrm{e}^{-\mathrm{tr}(AXBX')}, \tag{7.7}$$

where X is $m \times n$, $A = A' > 0, B = B' > 0$ are $m \times m$ and $n \times n$ positive definite constant matrices, a prime denoting a transpose. Then the Fourier transform of $g(X)$ is given by the following:

$$\phi_g(T) = E[\mathrm{e}^{i\mathrm{tr}(XT)}], \quad i = \sqrt{-1}, \tag{7.8}$$

where T is as in (7.6). Consider the transformation

$$Y = A^{\frac{1}{2}}XB^{\frac{1}{2}} \Rightarrow \mathrm{d}Y = |A|^{\frac{n}{2}}|B|^{\frac{m}{2}}\mathrm{d}X, \tag{7.9}$$

see Mathai (1997), where $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ denote the positive definite square roots of A and B respectively. Then simplifying the exponent, we have

$$\mathrm{tr}(A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}) - \mathrm{tr}(iXT) = \mathrm{tr}[(Y-C)(Y-C)'] - (\frac{i^2}{4})\mathrm{tr}(TA^{-1}T'B^{-1}),$$

where

$$C = \frac{i}{2}B^{-\frac{1}{2}}TA^{-\frac{1}{2}}.$$

Integrating out, we have

$$\phi_g(T) = \mathrm{e}^{-\frac{1}{4}\mathrm{tr}(B^{-\frac{1}{2}}TA^{-1}T'B^{-\frac{1}{2}})}. \tag{7.10}$$

Definition 1. (*Matrix-variate Linnik Density*) Let X and T be as defined in (7.10). Then X will be called a real rectangular matrix-variable Linnik random variable if its characteristic function is given by

$$\phi_X(T) = \mathrm{e}^{-[\mathrm{tr}(T\Sigma_1 T'\Sigma_2)]^{\frac{\alpha}{2}}}, \quad 0 < \alpha \leq 2, \tag{7.11}$$

where $\Sigma_1 > 0$ is $m \times m$ and $\Sigma_2 > 0$ is $n \times n$ positive definite constant matrices.

For the real rectangular matrix-variate Gaussian density of Example 1, $\Sigma_1 = \frac{1}{2}A^{-1}$ and $\Sigma_2 = \frac{1}{2}B^{-1}$ and $\alpha = 2$.

Theorem 7.1. *Let y be a real scalar random variable, distributed independently of the $m \times n$ matrix X and having a gamma density with the Laplace transform*

$$L_y(t) = (1 + \delta t)^{-\beta}, \quad \delta > 0, \beta > 0, \tag{7.12}$$

and let X be distributed as a real rectangular matrix-variate Linnik variable as given in Definition 1. Then the rectangular matrix $U = y^{\frac{1}{\alpha}}X$ has the characteristic function

$$\phi_U(T) = [1 + \delta(\mathrm{tr}(T\Sigma_1 T'\Sigma_2))^{\frac{\alpha}{2}}]^{-\beta}. \tag{7.13}$$

As special cases, we have the following corollaries:

Corollary 7.1. *When $m = 1, \Sigma_1 = I_1$, the characteristic function in (7.13) reduces to*

$$\phi_U(T) = [1 + \delta(T'\Sigma_2 T)^{\frac{\alpha}{2}}]^{-\beta}$$

which is the characteristic function of the multivariable Linnik variable with parameters $\delta, \beta, \Sigma_2 > 0$ defined in Lim and Teo (2009).

Corollary 7.2. *When $n = 1, \Sigma_2 = I_1$, then again $\phi_U(T)$ is the same as in Corollary 7.1 with the parameters $\delta, \beta, \Sigma_1 > 0$.*

COROLLARY 7.3. *When* $m = 1, \Sigma_1 = 1, \Sigma_2 = \mathrm{diag}(\sigma_{11}, .., \sigma_{nn}), \sigma_{jj} > 0, j = 1, ..., n$, *then* $\phi_U(T)$ *of (7.13) is given by*

$$\phi_U(T) = [1 + \delta(\sigma_{11}t_1 + ... + \sigma_{nn}t_n)^{\frac{\alpha}{2}}]^{-\beta}.$$

DEFINITION 2. The matrix variable U in (7.13) will be called a *real rectangular matrix-variate Gaussian Linnik variable*, when $\alpha = 2$ and Gamma Linnik variable for the general α.

By using the multi-index Mittag-Leffler function of Kiryakova [8], one can also define a *Gamma-Kiryakova vector or matrix variable.*

Now, let us consider two independently distributed real random variables y and X, where y has a general Mittag-Leffler density and X has a real rectangular matrix-variate Gaussian density as in (7.7). Let the density of y be given by

$$f_y(y) = \frac{y^{\alpha\beta-1}}{(\delta)^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k(\beta)_k}{k!(\delta)^k} \frac{y^{\alpha k}}{\Gamma(\alpha\beta + \alpha k)}, \quad y \geq 0,\, \delta > 0,\, \beta > 0. \tag{7.14}$$

Then we have the following result:

THEOREM 7.2. *Let* y *have the density in (7.14) and the real rectangular matrix* X *have the density in (7.7) and let* X *and* y *be independently distributed. Then* $U_1 = y^{\frac{1}{2}}X$ *has the characteristic function*

$$\phi_{U_1}(T) = [1 + \frac{\delta}{4^{\alpha}}(\mathrm{tr}(A^{-1}T'B^{-1}T))^{\alpha}]^{-\beta}. \tag{7.15}$$

THEOREM 7.3. *Let* $U_2 = y^{\frac{1}{\gamma}}X$ *where* y *and* X *are independently distributed with* y *having the density in (7.14) and* X *is real rectangular matrix-variate Linnik variable with the parameters* γ, A, B, *then* U_2 *has the characteristic function*

$$\phi_{U_2}(T) = [1 + \delta(\mathrm{tr}(A^{-1}T'B^{-1}T))^{\frac{\alpha\gamma}{2}}]^{-\beta}, \quad \delta > 0,\, \gamma > 0,\, \beta > 0,\, \alpha > 0. \tag{7.16}$$

This U_2 will be called a *Lévy Mittag-Leffler real rectangular matrix-variate random variable.*

Concluding remarks: The various Mittag-Leffler functions, and their generalizations to vector and matrix variable cases, discussed in this paper will be useful for investigators in various disciplines of applied sciences and engineering. The importance of Mittag-Leffler function in physics is steadily increasing. It is simply said that deviations of physical phenomena from exponential behavior could be governed by physical laws through Mittag-Leffler functions (power-law). Currently more and more such phenomena are discovered and studied.

It is particularly important for the disciplines of stochastic systems, dynamical systems theory, and disordered systems. Eventually, it is believed that all these new research results will lead to the discovery of truly nonequilibrium statistical mechanics. This is statistical mechanics beyond Boltzmann and Gibbs. This nonequilibrium statistical mechanics will focus on entropy production, reaction, diffusion, reaction-diffusion, etc and may be governed by fractional calculus.

Right now, the fractional calculus and the H-function (incl. the Mittag-Leffler function) are becoming very important in research in physics.

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